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LETTER TO THE EDITOR

On the stability of randomly frustrated systems with finite connectivity

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Abstract. We study dilute spin systems with finite connectivity, i.e. Viana–Bray random bond systems, near zero temperature. We give a stationary free energy in terms of a global order parameter g together with the equation of motion for g and the stability matrix. For the case where replica symmetry is *not* broken, we diagonalise the stability matrix around the ansatz given by Mézard and Parisi, and Kanter and Sompolinsky, close to the point of percolation ($\alpha = 1$) and for small admixtures of antiferromagnetic bonds. We find that the replica symmetric ansatz is *unstable* in the spin glass phase and in part of the ferromagnetic phase.

It has recently been shown that Ising spin glass systems have a replica symmetric stable spin glass phase for a certain range of non-Gaussian bond distributions (de Dominicis and Mottishaw 1986, 1987a, b, hereafter referred to as I, II and III). This gives rise to the possibility of a single 'state' spin glass phase in three dimensions. A d-dimensional nearest-neighbour spin glass with a non-Gaussian bond distribution is described in the zero-loop or tree approximation by a sequence of order parameters $q_{\alpha_1 \ldots \alpha_r} = \langle \sigma_{\alpha_1} \ldots \sigma_{\alpha_r} \rangle$, $r = 1, 2, \ldots, n$. In the Gaussian case $q_{\alpha\beta}$ alone is sufficient. A perturbative expansion valid close to the transition which includes only the first few order parameters ($r = 1, 2, 3, 4$) may sometimes be employed (Viana and Bray 1985, I, II, III). However, for certain ranges of temperatures and distributions the higher-order parameters become important and the perturbation expansion fails. One is thus forced to treat them all together by introducing a global order parameter g, for which in II and III we derived a generalised equation of motion. Orland (1985) and Mézard and Parisi (1985) had been previously led to take such a step in optimisation problems.

A special case of the problems considered in II is the Viana–Bray model, i.e. a model for dilute systems with finite connectivity. Here all pairs interact and the bond probability has the form

$$\mathscr{P}(J_{ij}) = \left(1 - \frac{\alpha}{N}\right)\delta(J_{ij}) + \frac{\alpha}{N} f(J_{ij}) \tag{1}$$

where α is the average connectivity of each site, N is the number of sites and $f(J_{ij})$ is normalised to unity, for example

$$f(J_{ij}) = a\delta(J_{ij} - J) + (1 - a)\delta(J_{ij} + J). \tag{2}$$

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In this model mean field theory is exact (instead of being the first step in, for example, a loop expansion when only nearest neighbours interact). In the low temperature, T, region, all $q_{\alpha_1\ldots\alpha_r}$ become important and one is forced to use the global order parameter g introduced in II. Formulae for the equation of motion of g (II(42), III(2.58)) apply to the bond distribution (1, 2). For the Viana-Bray model, Mézard and Parisi have derived an equivalent equation of motion (without using replicas, but for what amounts to a replica symmetric g). Ansätze solving the g equation of motion in the low T limit have recently been given by Mézard and Parisi (1987, hereafter referred to as MP) and Kanter and Sompolinsky (1987, hereafter referred to as KS). The low T limit is particularly important for the dilute graph partitioning problem introduced by Fu and Anderson (1986).

Here we present the following new results.

(i) A stationary free energy functional for the above Viana-Bray systems, in terms of a global order parameter $g\{\sigma_\alpha\}$, valid with or without replica symmetry breaking.

(ii) Its corresponding $(2^n \times 2^n)$ stability matrix. We give the form of the 2^n eigenvectors which describe fluctuations about the replica symmetric ansatz. In the low temperature region we use the MP and KS ansatz to explicitly obtain part of the spectrum of the stability matrix. This is sufficient to show that, in part of the ferromagnetic phase and in all of the spin glass phase, the MP-KS ansatz is unstable. We give the instability line explicitly near the percolation transition ($\alpha \sim 1$) and with a small admixture of antiferromagnetic bonds ($a \sim 1$). This instability makes it necessary to look for a symmetry breaking ansatz for which the formalism presented here is particularly useful.

The stationary free energy takes the form

$$-\beta f n = -\alpha/2 - [2\alpha(2\sinh 2\beta J)^n]^{-1} \operatorname*{Tr}_{\sigma_\alpha\tau_\alpha} g_n\{\sigma_\alpha\} g_n\{\tau_\alpha\}\left(a \prod_{\alpha=1}^{n} [\sigma_\alpha\tau_\alpha \exp(\beta J\sigma_\alpha\tau_\alpha)] \right.$$

$$\left. + (a-1) \prod_{\alpha=1}^{n} [-\sigma_\alpha\tau_\alpha \exp(-\beta J\sigma_\alpha\tau_\alpha)] \right) + \ln \operatorname*{Tr}_{\sigma_\alpha} \exp(g_n\{\sigma_\alpha\}) \qquad (3)$$

where $g_n\{\sigma_\alpha\}$ is the generalised order parameter, with 2^n components:

$$g_n\{\sigma_\alpha\} = \alpha \sum_{r=0}^{n} b_r \sum_{(\alpha_1\ldots\alpha_r)} q_{\alpha_1\ldots\alpha_r} \sigma_{\alpha_1} \ldots \sigma_{\alpha_r}$$

where

$$b_r = \begin{cases} \cosh^n \beta J \tanh^r \beta J & r \text{ even} \\ (2a-1)\cosh^n \beta J \tanh^r \beta J & r \text{ odd.} \end{cases}$$

When we restrict ourselves to the replica symmetric (RS) case, i.e. $g_n\{\sigma_\alpha\} = g_n(\hat\sigma)$ with $\hat\sigma = \Sigma_n \sigma_\alpha$ we get

$$-\beta f = (\alpha/2) \ln 2 \sinh 2\beta - \int_{-i\infty}^{+i\infty} ds\, dt \int_{-\infty}^{+\infty} \frac{du\, dv}{(2\pi i)^2} \exp(us + vt) g_0(s) g_0(t)$$

$$\times \{a \ln[2(e^{\beta J}\cosh(u+v) - e^{-\beta J}\cosh(u-v))]$$

$$+ (a-1)\ln[2(e^{\beta J}\cosh(u-v) - e^{-\beta J}\cosh(u+v))]\}[2\alpha(a-1)]^{-1}$$

$$+ \int_{-i\infty}^{+i\infty} ds \int_{-\infty}^{+\infty} \frac{du}{2\pi i} e^{us} \exp(g_0(s) - \alpha) \ln 2\cosh u. \qquad (4)$$

The equation of motion from the stationarity of equation (3) is (III, equation 2.58)†

$$g_n\{\sigma_\alpha\} = Z_g^{-1}\alpha \operatorname*{Tr}_{\tau_\alpha} \exp(g_n\{\tau_\alpha\})\left[a \exp\left(\beta J \sum_\alpha \sigma_\alpha\tau_\alpha\right) + (1-a)\exp\left(-\beta J \sum_\alpha \sigma_\alpha\tau_\alpha\right)\right] \quad (5)$$

where

$$Z_g = \operatorname*{Tr}_{\tau_\alpha} \exp(g_n(\tau_\alpha)). \quad (6)$$

In the RS case $g_n\{\sigma_\alpha\} = g_n(\hat\sigma)$, one gets (II, equation 42)‡

$$g_0(\hat\sigma) = \alpha\, e^{-\alpha}\int_{-i\infty}^{+i\infty} ds \int_{-\infty}^{+\infty} \frac{du}{2\pi i}\exp(g_0(s)+us)\{a \exp[\hat\sigma \tanh^{-1}(\tanh u \tanh \beta J)] + (1-a)\exp[-\hat\sigma \tanh^{-1}(\tanh u \tanh\beta J)]\}. \quad (7)$$

This global order parameter is related to the bond averaged distribution of the site magnetisation $P(m)$ used in the MP derivation by

$$g_0(\hat\sigma) = \alpha \int_{-1}^{+1} dm\, P(m)\{a \exp[\hat\sigma \tanh^{-1}(m \tanh \beta J)] + (1-a) \times [-\hat\sigma \tanh^{-1}(m\tanh\beta J)]\} \quad (8)$$

$$P(m) = \overline{\delta(m - \langle\sigma\rangle)}. \quad (9)$$

In the $\beta J \gg 1$ limit the ansatz of MP and KS is

$$g_0(\hat\sigma)/\alpha = 1 - Q + Q_+\, e^{\hat\sigma\beta J} + Q_-\, e^{-\hat\sigma\beta J} \quad (10)$$

$$Q_\pm = \tfrac{1}{2}Q \pm M(a - \tfrac{1}{2}) \quad (11)$$

with Q (fraction of frozen spins) and M (magnetisation of frozen spins) given by

$$1 - Q = e^{-\alpha Q} I_0(2\alpha(Q_+Q_-)^{1/2}) \quad (12)$$

$$M = \alpha\, e^{\alpha Q}\int_0^1 dv\, I_0[2\alpha(Q_+Q_-)^{1/2}(1-v)^{1/2}] \times [Q_+ \exp(\alpha Q_+ v) - Q_- \exp(\alpha Q_- v)] \quad (13)$$

with the corresponding free energy

$$-\beta f = \beta J\alpha[M^2(a-\tfrac{1}{2}) + \tfrac{1}{2}(1-Q)^2 + Q\, e^{-\alpha Q} I_0(2\alpha(Q_+Q_-)^{1/2}) + 2(Q_+Q_-)^{1/2} e^{-\alpha Q} I_1(2\alpha(Q_+Q_-))^{1/2}] + \ln 2[e^{-\alpha Q} I_0(2\alpha(Q_+Q_-))^{1/2} - \tfrac{1}{2}\alpha(1-Q)]^2. \quad (14)$$

The form of (13) is different from the one given by KS. For brevity we only give the solution of (12) and (13) close to $\alpha = a = 1$, i.e. $\alpha = 1+\varepsilon$, $a = 1 - \mu\varepsilon$. As in MP and KS we find three phases. To lowest order in ε:

(i) paramagnet $Q = M = 0$ for $\varepsilon < 0$ (15)

(ii) ferromagnet $Q = 2\varepsilon(1-2\mu)$, $M = 2\varepsilon[(1-2\mu)(1-6\mu)]^{1/2}$ for $\varepsilon > 0$, $\mu < \frac{1}{6}$ (16)

† With in III (2.58) $z = N$, $b_0 = 0$, $G_n\{\sigma_\alpha\} + \alpha \equiv g_n\{\sigma_\alpha\}$.
‡ With in II (42) $z = N$, $G(2it) + \alpha \equiv g(\hat\sigma)$. There we were interested in symmetric bond distribution, hence the cosine instead of the exponential.

(iii) spin glass $Q=4\varepsilon/3$, $M=0$ for $\varepsilon>0$, $\mu>\frac{1}{6}$. (17)

Taking the second derivative of (3) we get the eigenvalue equation

$$\operatorname*{Tr}_{\tau_\alpha}(\delta^2\beta f\{g_n\}/\delta g_n\{\sigma_\alpha\}\delta g_n\{\tau_\alpha\})f\{\tau_\alpha\}=\lambda f\{\sigma_\alpha\} \tag{18}$$

which we write as

$$\alpha^{-1}f\{\sigma_\alpha\}+(\alpha Z_g)^{-1}g_n\{\sigma_\alpha\}\operatorname*{Tr}_{\tau_\alpha}\exp(g_n\{\tau_\alpha\})f\{\tau_\alpha\}-\operatorname*{Tr}_{\tau_\alpha}(\lambda+Z_g^{-1}\exp(g_n\{\tau_\alpha\}))$$
$$\times\left[a\exp\left(\beta J\sum_\alpha\tau_\alpha\sigma_\alpha\right)+(1-a)\exp\left(-\beta J\sum_\alpha\tau_\alpha\sigma_\alpha\right)\right]f\{\tau_\alpha\}=0. \tag{19}$$

We have proved that in the RS case for $g_n\{\sigma_\alpha\}$, we can completely span the space of eigenvectors by 2^n functions of two variables

$$f\{\sigma_\alpha\}=f_{\{\mu_\alpha\}}(\hat{\sigma};\,q_{\sigma\mu}) \tag{20}$$

where

$$q_{\sigma\mu}=\sum_\alpha\sigma_\alpha\mu_\alpha. \tag{21}$$

Here $\{\mu_\alpha\}$ is a spin configuration that labels the eigenvector. All eigenvectors with the same $\hat{\mu}=\Sigma_\alpha\mu_\alpha$ are degenerate. Note that in the limit $\alpha\to\infty$, $J=J'\alpha^{-1/2}$ the model goes over to the Sherrington–Kirkpatrick (1975) model and we recover the results of de Almeida and Thouless (1978).

As $n\to 0$, (19) becomes an integral equation for the eigenvalues $\lambda_{\hat{\mu}}$ which we have solved in the relevant regions (ii) and (iii).

To identify the instability it is sufficient to consider the four $\hat{\mu}$ independent eigenvalues

$$\left(\frac{1}{\alpha}-\left(\lambda+A_0\mp\frac{(A_++A_-)}{2}\right)\right)\left(\frac{1}{\alpha(2a-1)}-\left(\lambda+A_0\pm\frac{(A_++A_-)}{2}\right)\right)$$
$$+\frac{(A_+-A_-)^2}{4}=0 \tag{22}$$

where

$$A_0=e^{-\alpha Q}I_0(2\alpha(Q_+Q_-))^{1/2}\qquad A_\pm=e^{-\alpha Q}(Q_+/Q_-)^{\pm1/2}I_1(2\alpha(Q_+Q_-))^{1/2}.$$

The eigenvectors corresponding to each eigenvalue include both longitudinal (replica symmetric) and transverse (replica symmetry breaking) vectors. We note that, for $a<\frac{1}{2}$ and $\alpha<1$, two of these eigenvalues are negative in the paramagnetic phase. The reason for this is not clear to us.

In the vicinity of $a=1$ and $\alpha=1$, the four eigenvalues take the form (see (15)–(17))

(i) paramagnetic phase ($\varepsilon<0$), all positive

(ii) ferromagnetic phase ($\varepsilon>0$, $\mu<\frac{1}{6}$)

$$\lambda_{1,2}=\{(1-3\mu)\pm[(1-3\mu)^2+(1-2\mu)(1-6\mu)]^{1/2}\}\varepsilon$$
$$\lambda_{3,4}=\{(1-3\mu)\pm[(1-3\mu)^2-(1-2\mu)(1-10\mu)]^{1/2}\}\varepsilon$$

(iii) spin glass phase ($\varepsilon>0$, $\mu>\frac{1}{6}$)

$$\lambda_1=\varepsilon$$
$$\lambda_2=2\varepsilon(\mu-\tfrac{1}{6})$$
$$\lambda_3=(1+2\mu)\varepsilon$$
$$\lambda_4=-\varepsilon/3.$$

We see that λ_1 and λ_3 are positive everywhere. The ferromagnet to spin glass transition is signalled by $\lambda_2 = 0$, but it remains positive in both phases. However λ_4 is negative for $\mu > \frac{1}{10}$, giving a line of instability in the ferromagnetic phase at $\mu = \frac{1}{10}$. Therefore the solutions (16) and (17) are unstable in part of the ferromagnetic phase and in all the spin glass phase.

The nature of the longitudinal (replica symmetric) eigenvectors corresponding to λ_4 indicate that the instability in this subspace might be removed by including a term $\int_{-\beta J}^{\beta J} \mathrm{d}K\, h(K)\, \mathrm{e}^{\hat{\sigma} K}$ in the MP–KS ansatz, equation (10).

The existence of transverse eigenvectors corresponding to λ_4 suggest that it will also be necessary to break replica symmetry to obtain a stable solution. The formalism presented here is particularly useful for this purpose.

In summary, we have shown that the ansatz proposed in KS and MP cannot describe the spin glass phase.

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